Fermi National Accelerator Laboratory

FERMILAB-Pub-80/88-THY
October 1980

How to Calculate the Heavy Quark Fragmentation Function--
An Application of Cut Vertices*

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*Presented as a thesis to the Department of Physics, The University of Chicago, in partial fulfillment of the requirements for the Ph.D. degree.

Abstract

I show that the heavy quark fragmentation function is calculable to leading order in the inverse heavy mass. This implies that the production of a heavy particle (with one heavy quark) with a given momentum is calculable. In first approximation the function is $\delta(1-x)$. The peak at $x=1$ is however not of zero width, but has a width on the order of the inverse mass. I determine the shape of the function for x near 1. Lastly I show that the spin of the heavy quark does not flip during its fragmentation. This paper extends the work of Mueller, Sterman and others.[1]

Introduction

The principal result of this paper is that the probability to produce a heavy particle (e.g. in e^+-e^-) with a given momentum is calculable. By heavy particle I mean one with a single heavy quark. The result is derived in the spirit of recent work[1] that develops a firm QCD basis, in perturbation theory, for the parton model. The approach is essentially theory independent. I give the explicit demonstration for ϕ_6^3 and for an abelian gauge theory.

In parton model language the statement is: 1) that heavy particles are produced only through the fragmentation of heavy partons, and 2) that a heavy parton fragments into a heavy particle with the same forward momentum. 1) and 2) are true up to corrections of order μ/M , where M is the heavy mass. μ is characteristic of the light hadronic scale, and is $\sim$1 Gev .

Thus the fragmentation function (FF) for a final heavy particle is $\sim\delta(1-x)$. This form was proposed some time ago, and several arguments for it have been given.[2,3]

The FF is a delta function only in first approximation--the peak at x=1 is actually not of zero width. I will derive the shape of the FF for x near 1 . It is pictured in fig. 13.

Lastly, I show that spin flips of the fragmenting heavy quark are suppressed. This determines the helicities with which spinning heavy particles are produced. If the quark is heavy enough, so that its weak decay is fast, then it will decay before its spin flips. The spin of the quark becomes directly observable. There is no longer sufficient time to produce a heavy hadron.

Since the correction to the main result is of order 1Gev/M , the result should apply to the bottom quark.

Sketch of the Argument

An explicit definition of the fragmentation function has been provided by Mueller.[4] His (time-like) cut vertices are essentially moments of the FF . His results give a parton model picture, expressed in the form of the Operator Product Expansion (OPE). To leading order, an inclusive production process, e.g. $e^+e^- \to H(p) + X$ (fig. 1), is factorized into a calculable function C -- a generalization of the Wilson's coefficient that represents the short time hard interaction of partons -- convoluted with the FF .

$$d\sigma_{e^+e^- \to H(p)+X}(Q^2, x=Q^2/2Q\cdot p, p^2) = \int_1^x dz/z \; C_{e^+e^- \to q_i}(Q^2,z) F_{q_i \to H}(x/z, p)$$

The FF, $F_{q_i \to H}$, generalizes the idea of an operator matrix element, with the fragmenting parton q_i corresponding to the operator, and the observed hadron to the state between which the operator is evaluated. It is defined so as to approximate the usual definition: the probability for $q_i(x/z\, p) \to H(p)+X$, over long times.

Mueller takes moments, and works always with the cut vertices. Since I need the FF itself, I will slightly rephrase his work (following Collins, Baulieu et al.[5])

My argument is based on kinematics. Imagine fragmentation as a physical process, with freely propagating on-shell partons interacting only at points. An initial light parton fragments in a cascade. Kinematics permit it to split into several collinear light fragments, which may then split further. The original forward momentum is divided among the final products. In contrast, an initial heavy quark (which cannot decay via the strong interaction) cannot lose any of its forward momentum. It is dressed by soft gluons into a heavy hadron with the same momentum.

The work lies in showing that Mueller's definition gives this physical picture of fragmentation. From one point of view this is a familiar task. The renormalization program can be considered as the demonstration that unphysically high virtual momenta do not contribute uncontrollably to low momentum processes. A better analogy is that of an operator renormalized and evaluated at a low momentum scale. The dominant contribution to the Feynman integral comes from low momentum flows. The unphysical region is cut off by subtracting counterterms.

Unphysical contributions to the FF are cut off in the same way. The FF as defined here is renormalized at the equivalent of the low momentum scale above. Counterterms serve again to eliminate the "large" momentum region. The result is that the dominant contributions to the FF come from physical configurations, to which the above kinematic argument applies.

In fact, all of the machinery developed to isolate and understand large momentum contributions to Feynman diagrams, in renormalization and in the OPE, can be taken over bodily as Mueller has done. Especially for ϕ^3_6 , the proof of factorization looks the same as that of the OPE. The demonstration that heavy partons do not decay into light particles is similar to the proof that the matrix element of a heavy operator between light states is suppressed.[6]

This is not the whole story. That the form of the proof is the same should not disguise the fact that kinematics for the OPE and for factorization differ drastically. For a complete proof, one must combine the old OPE type approach with the new understanding of the kinematics of a production process.[1]

The problem with a production process is that the particles produced are physical -- their momenta are time-like Minkowskian. For the OPE the Feynman diagram can be treated as though all momenta were Euclidean. In Minkowski space propagators are more likely to go on shell, making the Feynman integrand big and causing new large mass-dependent contributions to the Feynman integral. The dependence on large momentum becomes harder to separate out -- it threatens to become inextricably tangled up with that on small momenta and masses. This new IR type of contribution must be understood before one attempts to isolate the dependence on large momentum, as in factorization, or to estimate the contribution of an unphysical large momentum flow to a diagram, as I must do here.

A detailed understanding of the IR for Feynman diagrams describing production processes has been provided by Sterman and others.[1]

The results of ref. 1 imply that the IR contributions to the FF come only from subprocesses that realize the physical picture of fragmentation above. The unphysical part of the fragmentation process then has all its lines effectively far off shell. This virtual part of the FF <u>is</u> effectively virtual. It has no leading dependence on small momenta, but is scaled by large momenta. It can therefore be handled in the same way as are the large virtual momentum flows in the more familiar context of the OPE. Just as large momentum flows do not contribute to the operator matrix element, so is the unphysical contribution to the FF suppressed.

To summarize the main argument: I consider the FF for a heavy quark turning into a heavy hadron, $F_{h\to H}$. If these do not have the same forward momentum, then some part of the fragmentation must be unphysical -- in physical fragmentation the heavy quark retains its forward momentum. However I will show that unphysical contributions to the FF are suppressed. The FF can only be large for x near 1, where x is the ratio of the two forward momenta.

Summary of the Paper

I will discuss a ϕ_6^3 theory in part I, and an abelian gauge theory in part II. Like Mueller, I take all light partons to have mass $m \sim \mu$. In part II I work in the Feynman gauge.[4] The result is gauge invariant.

In section IA I define and renormalize the FF. The definition is based on the work of Mueller[4], and on ref. 5.

In IB I present the argument summarized above.

IC completes the proof of the main result. I show that a heavy (light) parton does not fragment into a light (heavy) hadron. I then use the Energy-Momentum sum rule to normalize the $F_{h\to H}$. I find that $F_{h\to H} = \delta(1-x)+O(\mu^2/M^2)$.

IIA defines and renormalizes the FF for an abelian gauge theory.

In IIB I apply the methods develop in ref. 8 to understand IR divergences. I briefly discuss one potential difficulty in the application of those methods to the FF.

Section IIC rederives the result of IB for an abelian gauge theory.

In IID I discuss the shape of $F_{h\to H}$ for x near 1. I again make use of the methods of ref. 8, especially the IR power counting procedure.

Section IIE investigates the shape of the heavy to light FF, $F_{h\to L}$, for $x>>1$. I show that the energy going into soft light particles from an initial heavy quark is small. (But it is bigger than the energy going into hard light particles.) From energy conservation I can then normalize $F_{h\to H}$, as for ϕ_6^3.

Lastly, in IIF, I recast the previous argument to take into account the spin of the heavy quark as it fragments.

IA IA -- Definition of the Fragmentation Function[4,5]: φ^3_6

The unrenormalized FF, F^{un}, is defined in terms of a cut Green's function $G(k,p)$, illustrated in fig. 2. In addition to the amputated final hadron lines, G has two external unamputated lines, one on either side of the cut. These lines represent the fragmenting parton and have momentum k. Then $F^{un} = k^+\int \frac{dk^- d^4\underline{k}}{(2\pi)^6}\ G(k^2,p^2,k\cdot p)$ where the factor k^+ is included to make F^{un} dimensionless and boost invariant.[5] I work in the infinite momentum frame[4], with $p = (p^+, m^2/2p^+ = p^-, \underline{0} = \underline{p})$.

F^{un} has UV divergences due to the integration of the initial hanging lines over arbitrarily large transverse momenta. For example fig. 3 has a logarithmic divergence:

$$\sim k^+\int dk^- d^4\underline{k}\ \frac{1}{k^2-m^2+i\varepsilon}\ \frac{1}{k^2-m^2-i\varepsilon}\ (-2\pi)\delta((k-p)^2-m^2)$$

$$\sim \frac{x}{2(x-1)}\int d^4\underline{k}\ \frac{1}{(\underline{k}^2/(x-1))^2}\ , \text{ with } x \equiv k^+/p^+ \ .$$

The FF therefore requires renormalization.

The renormalized FF may be written:[5]

$$F^{ren}(x) = \int_{1/x}^{1} \frac{d\eta}{\eta}\, k(\eta)\, F^{un}(\eta x) \ .$$

This is analogous to the renormalization of an operator O: $O^{ren} = ZO^{un}$.

k, like Z, is given by a sum of counterterms. Each counterterm is a divergent subdiagram evaluated with its external legs at the renormalization point p_τ. As for the matrix element of an operator, the divergent subdiagrams have just two external legs in addition to the initial unamputated lines. Since the divergence of such a subdiagram depends on the plus momentum in its external lines, p_τ^+ must be allowed to vary.

I chose the following renormalization convention:

$$p_\tau^2 = p_\tau^- = \underline{p}_\tau^2 = 0 \quad \text{for light external lines}$$

$$p_\tau^2 = M^2 \ , \ p_\tau^- = M^2/2p_\tau^+ \ , \underline{p}_\tau^2 = 0 \quad \text{for heavy external lines.}$$

A line with momentum p_τ is then near shell and collinear with the final hadron. The renormalization is physical in that particles with such momenta appear in a physical picture of the fragmentation process. The above convention is analogous to renormalizing an operator at a low momentum scale. As a result of this renormalization choice, unphysical configurations will be suppressed.

IB -- The Fragmentation Function for a Heavy Quark

I will show in this section that the fragmentation of a heavy quark to a heavy hadron is suppressed unless their plus momenta are the same. The relevant FF, $F_{h\to H}(k^+,p)$, is illustrated in fig. 4. I consider the case when $x = k^+/p^+ \gg 1$.

The first remark is that for $x \gg 1$ some part of the fragmentation is unphysical.

The propagators for the unamputated lines are

$$1/\ |k^2-M^2|^2$$

and $p=(p^+,M_H^2/2p^+,\underline{0})$.

M_H is the heavy hadron mass; $M_H \sim M$.

It must be true that:

$$(k-p)^2 \gtrsim 0$$

since the momentum crossing the cut is physical and time-like. But then:

$$(k^2-M^2) \gtrsim (x-1)M^2 \sim M^2 \qquad \text{for } x \gg 1 \quad .$$

This is the crucial kinematical result. The propagators are $\lesssim 1/M^4$, and F can be order one only if the integration over large transverse momenta ($\gtrsim$ M) compensates for this factor.

However F^{ren} is UV finite, so the integral over the large transverse momentum region is also suppressed. This is the reasoning of refs. 6 and 7.

Take the diagram of fig. 5 as an example. The final "hadron" is here an off shell heavy quark. I have

$$\sim k^+ \int dk^- \underline{d^4k} \; \frac{1}{|k^2-M^2|^2} \quad (\delta(k-p)^2-m^2) \doteq \delta((k-p_\tau(p^+))^2-m^2)$$

$$= \frac{x}{2(x-1)} \int \underline{d^4k} \quad \frac{1}{(\underline{k}^2/x-1 + X(M_H^2-M^2)+(x-1)M^2)^2} - \frac{1}{(\underline{k}^2/(x-1)+(x-1)M^2)^2}$$

$$\sim \frac{x(x-1)}{2} \quad \ln\left(\frac{X(M_H^2-M^2)+(x-1)M^2}{(x-1)M^2}\right) \sim \frac{(M_H^2-M^2)}{M^2}$$

Notice that the large $\underline{k}^2$ region is cut off by the subtraction.

I repeat the argument concerning the UV in more detail below. I examine the contributions of the possible large transverse momentum flows, and show that every one is suppressed.

For each such flow there is a corresponding subdiagram (the hard part) composed of the lines through which the large momentum flows. The contribution of a hard part, integrated over large transverse momenta of order Q $\gtrsim$ M, is, from power counting:[4]

$$\sim Q^{4-2E}$$

E is the number of amputated external lines of the hard part. 4-2E is the integrated hard part's dimension. The contribution of the rest of the diagram, integrated over small transverse momenta, is at most of order M^0.

The only contributions not suppressed by a power of $1/M^2$ are those for which the hard part has E=2. But a subdiagram with E=2 is UV divergent. F^{ren} contains a term that cancels this divergence, in which the hard part subdiagram is replaced by a counterterm. The counterterm is given by the same subdiagram, but with its external lines at the renormalization point.

The difference of the two terms integrated over the given momentum flow may be written:

$$k^+ \int_{\underline{k}^2 \sim Q^2} dk^- d^4\underline{k} \int_{\underline{r}^2 << Q^2} d^6 r \left\{ H(k,r) - H(k, p_\tau(r^+)) \right\} S(r,p) \quad .$$

This is illustrated in fig. 6.

H(k,r) is the hard part integrated over internal transverse momenta of order Q with the indicated external momenta. S represents the rest of the diagram, integrated only over small transverse momenta. If the initial propagators of S are each $\sim 1/M^2$ then $S \lesssim 1/M^4$. If they are each $\sim 1/\mu^2$, then the initial lines of S are heavy, and $r^+/p^+ - 1 \sim \mu^2/M^2$, $\underline{r}^2 \sim \mu^2 << M^2$.

Then

$$k^+ \int_{\underline{k}^2 \sim Q^2} dk^- d^4\underline{k} (H(k,r) - H(k, p_\tau(r^+)) \sim (r - p_\tau)^2/Q^2 \sim \mu^2/Q^2 \quad .$$

The contribution of momentum flows with E=2 is also suppressed. It follows that $F_{h \to H}(x) \sim \mu^2/M^2$ for $x >> 1$.

IC Normalization of $F_{h\to H}$

Implicit in the above was the argument that the fragmentation of a light parton to a heavy hadron is suppressed by μ^2/M^2. The demonstration that a heavy parton does not fragment to a light hadron is similar.

If the light hadron has a non-vanishing fraction w of the plus momentum of the initial parton--$w=1/x\gg\mu/M$--then from kinematics the initial propagators are far off shell. From power counting, the only momentum flows that might redress the propagator suppression are those for which the hard subdiagram has two light external legs. These flows are cancelled as before by counterterms.

Thus

$$F^{ren}_{\ell\to H}(x) \sim \mu^2/M^2 \;,\; F^{ren}_{h\to L}(x) \sim \mu^2/M^2 \quad \text{for} \quad x\ll M/\mu \;.$$

I have shown that $F_{h\to H}$ is suppressed for $x\gg 1$. Thus the FF may be approximated by:

$$F_{h\to H}(x) \sim C\delta(1-x)$$

C can be determined by means of the energy momentum sum rule:[4,9]

$$\int_0^1 \frac{dw}{w}\, w^5 \left[F^{ren}_{h\to H}(w) + F^{ren}_{h\to H}(w)\right] = 1 \;.$$

This expression is just the statement of energy conservation. The integrand corresponds to $F(p_z)E\,\frac{d^5p}{2E}$, where

$E,p \sim w$ are the energy and momentum of the final hadron.

From arguments that I will postpone to the next section, the behavior of $F_{h\to L}(w)$ for $w\ll 1$ is such that:

$$\int_0^1 \frac{dw}{w}\, w^5\, F_{h\to L}(w) \sim \mu^2/M^2 \;.$$

Thus $C = 1 + O(\mu^2/M^2)$, $F_{h\to H}(x) \cong \delta(1-x)+O(\mu^2/M^2)$.

This implies that $\int_0^1 \frac{dw}{w}\, w^4 F_{n\to H}(w) = 1$. The average number of heavy hadrons produced by a heavy parton is one, as expected.

II - Abelian Gauge Theory

I will now derive the above results for an abelian gauge theory. The approach, except for technical complications, is the same as for ϕ_6^3 . I work in the Feynman gauge.

IIA - The Fragmentation Function in an Abelian Gauge Theory

The FF is defined in terms of the cut Green's functions G_a^n and G_b, shown in fig. 7. Since I am interested in the fragmentation of heavy quarks, I will not discuss 7b, which represents the fragmentation of a gluon.

$G_a^n(r,k_1\ldots k_L;k_{L+1}\ldots k_n,r';p)$ has two external unamputated fermion lines and n unamputated gluon lines. The final hadron lines of G_a^n are amputated. The usual spin sum factor (e.g. $\not{p}$+M) is included on these lines if the hadron is not scalar.

Each external gluon line attaches at a γ^+ vertex in the body of G. The external fermion lines, one on either side of the cut, are joined at a γ^+ vertex.

The fermion unrenormalized FF is then defined by:

$$F_n^{un}(s^+,p) = \int\frac{d^4r}{(2\pi)^4}\prod_{i=1}^{n}\left[\left(\frac{-g}{k_i^+}\right)\frac{d^4k_i}{(2\pi)^4}\right]\frac{d^4r'}{(2\pi)^4}\; s^+(\delta(s^+-r^+-\sum_i^L k_i^+)G_a^n(r,k_i\ldots k_n,r')$$

g is the coupling constant.

The external momenta on the left (right) of the cut are incoming (outgoing).

This F_n^{un} is not gauge invariant. The gauge invariant FF is obtained by summing over all n: $F_{inv}^{un} = \sum_{n=0}^{\infty} F_n^{un}$.

The divergent subdiagrams of the FF are pictured in fig. 8. From power counting, one finds that they can have at most two external amputated fermion lines. But such diagrams may have in addition an arbitrary number of plus polarized external gluons.[4] (A plus polarized gluon attaches at a γ^- vertex.)

Alternatively, a divergent subdiagram may have just two external gluon lines (fig. 8b).

The divergence of the first type of subdiagram may be related by gauge invariance to that of a subdiagram with only two fermion external lines.[4] This is shown in fig. 9.

$$\mathrm{Div}\left\{ G(p,k_1 \ldots k_n,p')\right\} = \prod_1^n \left(\frac{-g}{k_i^+}\right) \mathfrak{D}\mathrm{iv}\left\{ G'(p,p')\right\}$$

G is the sum over a gauge invariant set of subdiagrams. G' is the same sum, but without the external gluons.

In renormalizing, then, counterterms may be defined in terms of subdiagrams with just two external lines. As before counterterms are evaluated with the external lines at momentum p_τ .

For light fermion external lines, a $\gamma^- = p_\tau^+\gamma^-/p_\tau^+ \sim \not{p}_\tau\ /p_\tau^+$, simulating a spin sum factor, is included in the definition of the counterterm.[4] For heavy fermion lines, a factor $(\not{p}_\tau+M)/p^+$ is used instead. Thus if a divergent subdiagram with two amputated external heavy lines is $\mathfrak{D}_{\alpha\beta}(s^+,p)$, the corresponding counterterm is

$$\frac{1}{4}\,\mathrm{tr}(\not{p}_\tau(p^+)+M)\,\mathfrak{D}(s^+,p_\tau(p^+))/p^+$$

IIB - $F_{h\to H}$ and IR Divergences

I will now treat the fragmentation of a heavy quark to a heavy hadron. The approach is complementary to that of the first part. I will again consider the contributions of different momentum flows to the Feynman integral, but the emphasis will be on the soft part of the flow rather than the hard. For a given momentum flow the soft part is roughly the largest cut subdiagram that contains the hadron lines and that has all external lines near shell and nearly collinear.

I investigate the soft part by means of the methods developed in refs. 8,1. These papers study the appearance of IR divergences in massless cut Green's functions. An IR divergence is associated with a reduced diagram that can be considered a space time picture of a physical process.

Here I wish to determine the soft parts that make order M^0 contributions to the FF. If the light mass scale were taken to zero these soft parts would give rise to IR divergences. The M^0 contribution of the soft part is due to reduced diagram configurations that are physical in the $m\to 0$ limit.

The leading reduced configurations for $F^{un}_{h\to H}(x), x>>1$, are as shown in fig. 10.

An on-shell heavy quark emerges from a contracted vertex and is dressed by zero momentum gluons into a heavy hadron. These soft gluons may attach anywhere along the heavy lines. The initial contracted vertex represents the hard part. Its lines are effectively far off shell: $p^2-m^2 \gtrsim M^2$.

There is one important subtlety in the derivation of the above conclusions. The analysis of ref. 8 dealt with cut Green's functions that corresponded to physical processes, such as $e^+e^-\to \text{jet}_1 + \text{jet}_2 +$ anything. The massless FF seems to have new types of IR divergences due to the $1/k^+$ factors associated with external gluons. This possibility is eliminated by a simple extension of the previous arguments.[10] One must choose the correct $i\epsilon$ prescription in defining the $1/k^+$ factor. The new type of divergence cancels between diagrams in which the initial gluons are on different sides of the cut.

IIC - Suppression of $F_{h\to H}$ for $x \gg 1$

It is now possible to show the suppression of $F_{h\to H}$ for $x \gg 1$. The dominant momentum flows are as shown in fig. 10. The hard part, or the initial contracted vertex, has just two external lines. There is therefore a counterterm graph corresponding to it in F^{ren}. Since F^{ren} is UV finite, the Q^0 behavior of the graph and countergraph cancels between them, when $Q \gg M$. Here Q characterizes the transverse momenta in the hard part and counterterm subdiagram. This cancellation occurs because the leading behavior of the subdiagrams is proportional to γ^+. The definition of the counterterm C is:

$$C = \frac{1}{4p_\tau^+}\,\mathrm{tr}(\not{p}_\tau + M)\,\mathcal{D}(p_\tau) \cong \frac{1}{4p_\tau^+}\left\{\mathrm{tr}(\not{p}_\tau + M)\gamma^+\right\}\mathcal{D}'(p_\tau)$$

$$\cong \mathcal{D}'(p_\tau) + O(M^2/Q^2)$$

where $\mathcal{D}(p_\tau)$ is a divergent subdiagram integrated over transverse momenta $\sim Q$ and $\mathcal{D}'$ is the divergent scalar coefficient of the γ^+ term of $\mathcal{D}$. The difference of graph and countergraph for the momentum flow under consideration is:

$$\int_{k^2 \ll Q^2} d^4k\,\left[\mathcal{D}(k) - C(p_\tau(k^+))\,\gamma^+\right] S(k,p) \cong \int_{k^2 \ll Q^2} d^4k\,\left[\mathcal{D}'(k) - \mathcal{D}'(p_\tau(k^+))\right]\gamma^+ S(k,p)$$

By the same reasoning as for ϕ^3_6 this is $\sim M^2/Q^2$ for $Q^2 \gg M^2$.

When $Q \sim M$ then $\mathrm{tr}\, \gamma^- D(p_\tau) \sim \mathrm{tr}\left(\frac{\gamma^+ \not{p}_\tau^-}{p_\tau^+} D\right) = \mathrm{tr}(\gamma^+ D)\frac{M^2}{2(p_\tau^+)^2}$

$$\sim \mathrm{tr}\left(\frac{M}{p_\tau^+}\right) D \sim M^0$$

For the leading M^0 behavior to cancel between graph and countergraph it must be true that $S \sim \frac{\not{p}_\tau + M}{p_\tau^+} + O(\mu/M)$. If this is so then the difference is:

$$\int_{r^2 \sim \mu^2} d^4r \ \left(D(r) - C(p_\tau)\gamma^r \right) S(r,p)$$

$$\sim \int_{r^2 \sim \mu^2} d^4r \ \mathrm{tr}\left\{ \left[\frac{\not{p}_\tau(r^+) + M}{r^+} \right] \left\{ D(r) - D(p_\tau(r^+)) \frac{1}{4} \mathrm{tr}\left[\gamma^+ \frac{\not{p}_\tau + M}{r^+} \right] \right\} \right\} S'(p,r)$$

$$\sim \int_{r^2 \sim \mu^2} d^4r \ \mathrm{tr}\left[\frac{\not{p}_\tau + M}{r^+} \left(D(r) - D(p_\tau) \right) \right] S'(p,r)$$

where S' is the coefficient in S of $(\not{p}_\tau + M)/r^+$

The M^0 behavior of the hard part cancels: $\mathrm{tr}\frac{(\not{p}_\tau + M)}{r^+}\left(D(r) - D(p_\tau)\right) \sim \mu^2/M^2$.

I must therefore show that the soft part S is of the prescribed form. In the important soft configurations the heavy quark interacts with soft lines at 3 point vertices.[8] In fact it has just coulomb interactions. This is so because in the hadron rest frame the heavy quark is non-relativistic. The strength of a spin interaction is proportional to the velocity imparted by it to the heavy quark, and thus is small.

The numerator at a soft interaction vertex is of the form:

$$(\not{p} + M)\gamma_\mu(\not{p} + \not{k} + M) = 2p_\mu \, \not{p} + M + O(k, p^2 - M^2)$$

k and p^2-M^2 are small since the gluon is soft and the heavy quark is near shell in the soft part. Thus a gluon interaction preserves the $\not{p}$+M structure of the numerator. The soft part is as desired.

Corrections to the $\not{p}$+M behavior of the soft part are suppressed by a factor μ/M due to a single spin interaction. Therefore $F_{h\to H}(x) \sim \mu/M$ for $x \gg 1$.

Similarly, for a light parton turning into a heavy hadron, $F_{\ell \to H}(x) \sim \mu/M$

.

Finally I consider $F_{h\to L}(x)$ for x not too large, i.e. $w = 1/x \gg M/M$.

In this case the dominant soft parts are as shown in fig. 11. They represent the unfolding of a jet of light particles. Again the hard contracted vertex part has a counterterm. The hard subdiagrams, with momenta $\gtrsim M$, to leading order are proportional to γ^+. Thus the M^0 behavior cancels between term and counterterm.

The correction to the leading behavior of the hard subdiagrams is $\sim \mu^2/M^2$ and

$$F_{h\to L}(x) \sim \mu^2/M^2 \text{ for } w \gg \mu/M$$

As an example of the suppression for $F_{h\to H}$, consider the FF diagram for fig. 12.

Fig. 12a is:

$$-i\,\delta(1-x)\int d^4k\ \frac{p^+}{k^+}\ \frac{\gamma^+(\not p + \not k + M)\gamma^+}{(p+k)^2 - M^2 + i\varepsilon}\ \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$= \delta(1-x)\int dk^+ d^2\underline{k}\quad 2\pi\left[\theta(p^+ + k^+) - \theta(k^+)\right]$$

$$\left\{\frac{p^+}{k^+}\ \frac{(p+k)^+}{k^+}\ \frac{1}{\left[\underline{k}^2\, p^+/k^+ + \frac{p^+ + k^+}{p^+}(p^2 - M^2) + \frac{k^+}{p^+}M^2\right]}\right\}$$

where x is the usual ratio of initial and final plus momenta.

Fig. 12b is:

$$\frac{(p^+ + k^+)^2}{k^+}\int d^2\underline{k}\ \left[-2\pi\,\theta(k^+)\right]\Big/\left[\underline{k}^2\, p^+/k^+ + (p^+ + k^+)/p^+\,(p^2 - M^2) + \frac{k^+}{p^+}M^2\right]$$

Here $x \equiv (p^+ + k^+)/p^+$

Including the counterterms and integrating over $\underline{k}$ yields approximately (for $|k^+| > \varepsilon$):

Fig. 12a: $$\sim \int_{-p^+}^{-\varepsilon} \left[-2\pi(p^+ + k^+)/k^+\right] \frac{(p^+ + k^+)(p^2 - M^2)}{k^+ M^2} \qquad \delta(1-x)$$

Fig. 12b: $$2\pi\, \theta(k^+) \frac{(p^+ + k^+)^2}{p^+ k^+} \frac{p^2 - M^2}{M^2}$$

When k^+ is not small, each diagram is suppressed. Upon adding the two, and integrating over their initial plus momenta, one obtains:

$$2\pi\, p^{+2} \frac{(p^2 - M^2)}{M^2} \left[-\int^{-\varepsilon} \frac{dk^+}{(k^+)^2} + \int_{\varepsilon} \frac{dk^+}{(k^+)^2} \right] \sim \frac{\mu^2}{M^2} \quad \text{as } \varepsilon \to 0$$

Thus the fig. 12 contribution to the FF is suppressed as expected, since there is no dominant soft configuration of the fig. 10 type.

IID - The Shape of $F_{h\to H}$

I now consider the behavior of $F^{\text{ren}}_{h\to H}(x)$ for $x \to 1$. I have shown that $F_{h\to H} \sim \mu/M$ for $x \to 1$ and that this is due to an order μ/M correction to the soft part. The contribution of the hard part, since it is incompletely cancelled, is order one for this leading behavior. The soft part is approximately a delta function $\delta(\ell^+ - p^+)$, where ℓ^+ (p^+) is its initial (final) plus momentum. It is therefore independent of x. The hard part remains order one for all x, including x near 1. Thus $F_{h\to H}$ has a contribution that is of order μ/M independent of x.

At some point other types of contributions will become bigger than μ/M . As x goes to 1 essentially the whole of the FF comes from reduced configurations in which the initial heavy line is close to shell and dressed softly into the hadron. They make $F_{h\to H}$ order one for x close enough to 1. There is no unphysical part and no suppression.

As long as $x-1 >> \mu^2/M^2$ the argument showing suppression holds good. One can still divide the fragmentation into hard and soft parts. But the hard part, unlike the soft, does depend on x , and the suppression of its contribution is weakened as $x \to 1$.

I will therefore determine the leading behavior in $1/(x-1)$ of the hard part as $x \to 1$, but with $x-1 >> \mu^2/M^2$.

Let $H(r)$ denote a hard subdiagram integrated over transverse momenta $\gtrsim M$ with two heavy external lines at momentum $r \sim p_c(r^+)$. Define

$$H'(r) = \frac{1}{4}\, tr\left[\frac{(\not{r}+M)}{r^+} H(r) \right]$$

Note first that

$$H'(r) = H_0(r) + O(\mu^2/M^2)$$

$H_0(r)$ is $H'(r)$ with all light masses set to zero. The above follows since $H(r)$ is off shell.

The M^0 behavior of H is cancelled by the subtraction of a counterterm. The difference is:

$$H'(r) - H'(p_c(r^+)) \cong (r^- - p_c^-)\, \frac{d}{dr^-} H_0(r)\Big|_{p_c} + (\underline{r} - \underline{p}_c)\cdot \frac{d}{d\underline{r}} H_0(r)\Big|_{p_c} + O(\mu^2/M^2)$$

If the soft part $S(r,p)$ is not suppressed then, as assumed in the above expression, $r^- - p_c^-(r^+) << M^2/r^+,\ \underline{r}^2 << M^2$. The leading large scale dependence of the difference can be approximated by taking $r - p_c$ to be infinitesimal and replacing subtraction by differentiation.

The leading reduced configurations cause an IR divergence of this differentiated object at x=1 . The leading behavior in 1/(x-1) as $x \to 1$ is just the strength of this divergence. It can be established by the IR power counting technique of refs. 8, 9.

The divergence in Ho integrated down to x=1 is logarithmic, by these methods. The derivatives give an additional heavy propagator going on shell as $x \to 1$. The extra propagator changes the strength of the divergence from logarithmic to linear. Only the term of the FF involving the difference of a divergent hard subdiagram and its counterterm has this linear rise.

The behavior determined above is:

$$\int_{1+\varepsilon} dx\, F^{ren}_{h\to H}(x) \sim \frac{1}{\varepsilon}\mu^2/M^2 \qquad F^{ren}_{h\to H}(1+\varepsilon) \sim \frac{1}{\varepsilon^2}\mu^2/M^2$$

It becomes dominant when $\mu/M < \frac{1}{\varepsilon^2}\mu^2/M^2$ or $x-1 < \sqrt{\mu/M}$

Eventually the $1/(x-1)^2$ behavior of $F_{h\to H}$ is cut off. For x-1 small enough it becomes a bad approximation to replace the difference of H and its counterterm by a derivative. Clearly it must hold that $H(p) - H(p_\gamma) \lesssim H(p_\gamma)$ or $1/\varepsilon^2\, \mu^2/M^2 \lesssim 1/\varepsilon$. Thus the $1/(x-1)^2$ is cut off roughly when $x-1 \sim \mu^2/M^2$.

The shape of $F_{h\to H}$ is pictured in fig. 13.

Lastly, notice that the $1/(x-1)^2$ behavior of $F_{h\to H}(x)$ comes from a part of the fragmentation which is hard. Momenta squared in this part are(from the power counting argument) typically $\gtrsim (x-1)M^2$, which is $\gg \mu^2$ in the region where one expects to see this behavior. Thus for a nonabelian gauge theory, asymptotic freedom should apply. The hard part should be given, up to logarithmic corrections $1/\ln\left((x-1)M^2/\mu^2\right)$ by the lowest order graph, with the coupling constant g renormalized at the scale $(x-1)M^2$.

IIE - Normalization of $F_{h \to H}$

I wish in this section to derive the overall normalization of F using the energy-momentum sum rule. In doing so I will need to know the total average energy going into light particles from a fragmenting heavy parton:

$$\int_0^1 F_{h \to L}(w)\, w^2 \frac{dw}{w}$$

Therefore, I will now discuss the behavior of $F_{h \to L}(w)$ as w becomes small.

As in the last section, the argument for suppression and the division of the fragmentation into hard and soft parts work as long as $w \gg \mu/M$.

Unlike the previous case, the soft part is order one independent of its initial or final plus momentum. The contribution of the contracted vertex is $\sim \mu^2/M^2$, but its suppression is weakened as its external legs carry off less and less of the starting plus momentum.

As $w \to 0$, there is an order one contribution to $F_{h \to L}$ due to configuration like that in fig. 14. The reduced diagram shows the heavy quark propagating with unchanged momentum across the cut. It interacts with a soft cloud that includes the final hadron lines.

It is the influence of such a configuration that weakens the hard part's suppression.

Consider therefore $H(r)$, a subdiagram with two light external fermion legs . (The treatment of the gluon case is similar.) Define H'(r) = $\frac{1}{4}\mathrm{Tr}\gamma^- H$. As before:

$$H'(r) = H_0(r) + O(\mu^2/M^2)$$

For w not too small:

$$H(r) - H(p_\tau) \simeq (r^- - p_\tau^-)\frac{d}{dp^-} H_0(r)\Big|_{p_\tau} + (\underline{r} - \underline{p}_\tau)\cdot\frac{d}{d\underline{r}} H_0(r)\Big|_{p_\tau}$$

I ask again what is the strength of the IR divergence in this differentiated object as $w \to 0$, as a dominant configuration is approached.

Take $Ho(p_\tau)$ first. Normally a configuration with a heavy quark and soft cloud would give, by power counting, a logarithmic divergence. This is modified here because the external lines are included in the soft cloud. Normally their momentum would be integrated, but is not here. Also the mass shell delta function and numerator momentum factor associated with an ordinary cut line are missing.

These changes imply that $Ho(p_\tau) \sim \frac{1}{w^3}$ as $w \to 0$.

The differentiated object behaves like $\frac{d}{dp^-} Ho(p)\big|_{p_\tau} \sim 1/w^4$. As before, the derivative adds an extra propagator.

Finally, recall that $p_\tau^- = \frac{m^2}{2p^+} \sim 1/w$. Therefore

$$H(p) - H(p_\tau) \sim \frac{1}{w^5} \mu^2/M^2 \quad \text{as} \quad w \to 0$$

This behavior will be cut off when $1/w^5 \, \mu^2/M^2 \sim 1/w^3$, or when $w \sim \mu/M$.

The energy-momentum sum rule is:

$$1 = \int_0^1 \left[F_{h\to L}(w) + F_{h\to H}(w) \right] \frac{w^3 dw}{w}, \quad \int_0^1 F_{h\to L}^{u\,n}(w) \frac{w^3 dw}{w} \sim \int_0^1 [H'(w)w] \frac{w^3 dw}{w}$$

The extra w occurs on the right to make H' a physical probability to produce a light fermion. Recall that H was defined by tracing with a γ^-. What is needed here is a trace with the usual final particle factor $\not{p} \sim p^+\gamma^- \sim w\gamma^-$.

Then

$$\int_\varepsilon^1 F_{h\to L}(w) \frac{w^3 dw}{w} \sim \frac{\mu^2}{M^2} \int_\varepsilon^1 \frac{dw}{w^2} \sim \frac{1}{\varepsilon} \mu^2/M^2$$

As above $\varepsilon \sim \mu/M$. For $w \lesssim \varepsilon$, $F_{h\to L}(w) \lesssim 1/w^2$

Thus:

$$\int_0^1 F_{h\to L}(w) \frac{w^3 dw}{w} \sim \mu/M$$

$$\int_0^1 F_{h\to H}(w) \frac{w^3 dw}{w} = 1 + O(\mu/M)$$

Again, the $1/w^5$ behavior of $F_{h\to L}(w)$ is due to a hard part of the fragmentation, where momenta squared are typically wM^2. For an asymptotically free theory, the coefficient of the [illegible] μ^2/M^2 behavior should be asymptotically calculable from the lowest order graphs for H .

$F_{h\to L}(w)$ is shown in fig. 15.

As an illustration of the above general argument consider the fragmentation of a heavy quark to a final light scalar particle, shown in fig. 16. Fig. 16 and its counterterm diagram give:

$$\sim -k^+ \int dk^- d^2\underline{k} \quad \frac{1}{|k^2 - m^2|^2} \Big(\mathrm{Tr}\{\gamma^+ (\not{k} + M)(\not{k} - \not{p} + M)(\not{k} + M)\} \delta((k-p)^2 - M^2)$$

$$- \mathrm{Tr}\{\gamma^+ (\not{k} + M)(\not{k} - \not{p}_c + M)(\not{k} + M)\} \delta((k - p_c)^2 - M^2) \Big)$$

$$= -\frac{x(x-1)}{2} k^+ \int d^2\underline{k} \left\{ \frac{16M^2 - 4p^2}{(\underline{k}^2 + M^2 + x(x-1)p^2)^2} - \frac{16M^2}{(\underline{k}^2 + M^2)^2} \right.$$

$$\left. + \frac{4}{x(x-1)} \left[\frac{1}{\underline{k}^2 + M^2 + x(x-1)p^2} - \frac{1}{\underline{k}^2 + M^2} \right] \right\}$$

$$= -k^+ \frac{x(x-1)}{2} \pi \left[\frac{(-p^2)(16x(x-1) + 4)}{M^2 + x(x-1)p^2} + \frac{4}{x(x-1)} \ln M^2/(M^2 + x(x-1)p^2) \right]$$

For $w = 1/x \ll 1$, but $w \gg \mu/M, \mu^2 = p^2$, this is:

$$\sim \frac{k^+ \pi}{2w^2} \frac{(+\mu^2 16)}{w^2 M^2}$$

The average total energy carried off by light scalar particles is:

$$\int_0^1 F_{h\to S}(w) \frac{w^3 dw}{w} \sim \int_{\mu/M}^1 \frac{dw}{w^2} \left(+ \frac{\mu^2}{M^2} \right) \sim \mu/M$$

IIF - The Heavy Quark Spin

As remarked in IIC, the spin of the heavy quark does not flip in the soft physical part of the fragmentation. I have shown that physical configurations make the dominant contribution to the FF, so that the quark does not flip at all while fragmenting. Each spin flip is suppressed by a factor μ/M.

I can take advantage of this by rephrasing the original discussion. I will rewrite:

$$d\sigma_{e^+e^- \to H(p)+X}(Q^2, x, p^2) \cong \int_{1/x}^{1} d\eta/\eta \; C(Q^2, \eta) F(\eta x)$$

$$= \int_{1/x}^{1} d\eta/\eta \left\{ C_{e^+e^- \to h\uparrow} F_{h\uparrow \to H} + C_{e^+e^- \to h\downarrow} F_{h\downarrow \to H} \right.$$

$C_{e^+e^- \to h\uparrow}$ is now the probability to produce an on shell heavy quark with a particular helicity. $F_{h\uparrow \to H}$ describes the fragmentation of such a quark to the hadron.

This is realized by defining C and counterterms with a spin projection matrix. A factor $(1 \pm \gamma_5 \not{S})(\not{p}_\eta + M)/p_\eta^+$ is included on the external lines instead of $(\not{p}_\eta + M)/p_\eta^+$.

$F^{un}_{h\uparrow \to H}$ is defined in the same way as $F_{h \to H'}$ except that an initial $\gamma^+ (1+\gamma_5)/2$ ($\frac{1-\gamma_5}{2}$ for the other spin) replaces the initial γ^+. This determines then the initial helicity of the fragmenting quark. Also, I no longer insist that a spin sum for the final hadron be performed.

The renormalized $F^{ren} \equiv \begin{pmatrix} F_\uparrow^{ren} \\ F_\downarrow^{ren} \end{pmatrix}$ satisfies:

$$F^{ren} = \int_{1/x}^{1} K(\eta)\, F^{un}(\eta x)\, d\eta/\eta$$

where $K(\eta)$ is a two by two matrix. (I ignore the mixing with gluons, since it is not relevant.)

The finiteness of F^{ren} and factorization can be shown as before. Similarly one shows that $\vec{F} \sim \delta(1-x)\, \vec{V}$, with $\vec{V}$ a constant vector. However because the spin of the quark does not flip, $F_{h\uparrow \to H}$, $F_{h\downarrow \to H}$ are not suppressed only when the heavy quark in the final state H has the same helicity as the fragmenting quark.

The production of a final state with a given spin for the heavy quark is given approximately by the Wilson coefficient for the production of a heavy quark with that spin. Corrections to this result are of order μ/M.

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Figure 1: $G(Q^2, p^2, x = Q^2/2q\cdot p)$, a cut Green's function describing the process $\gamma^*(Q) \to H(p) + X$. Slashed lines are amputated.

Figure 2. A cut Green's function used in the definition of the FF

Figure 3. A diagram for $F^{un}(k^+, p)$

Figure 4. $F_{h\to H}(k^+, p)$. A double line is heavy.

Figure 5. A diagram and counterterm diagram for F^{ren}

Figure 6. The cancellation of the leading M^2 behavior of an unphysical contribution to F^{ren}

Figure 7. Cut Green's functions used in the definition of the FF for an abelian gauge theory.

Figure 8. Divergent subdiagrams of the unrenormalized fermion FF.

Figure 9. Divergent subdiagrams related by gauge invariance.

Figure 10. Leading configuration for $F^{un}_{h\to H}$. All lines in the soft blob have near zero momenta.

Figure 11. Leading configuration for $F^{un}_{h\to L}$. All lines in the reduced diagram are near shell and either collinear with the final hadron or at near zero momentum.

Figure 12. Diagrams for $F_{h\to H}$

Figure 13. $F^{ren}_{h\to H}(x)$

Figure 14. A leading configuration at w=0 for $F_{h\to L}$. Lines in the soft blob have near zero momenta.

Figure 15. $F^{ren}_{h\to L}(w)$

Figure 16. Example of a diagram for $F_{h\to L}$

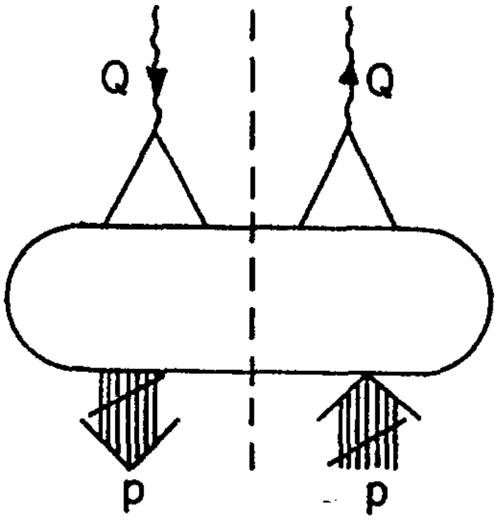


Fig. 1

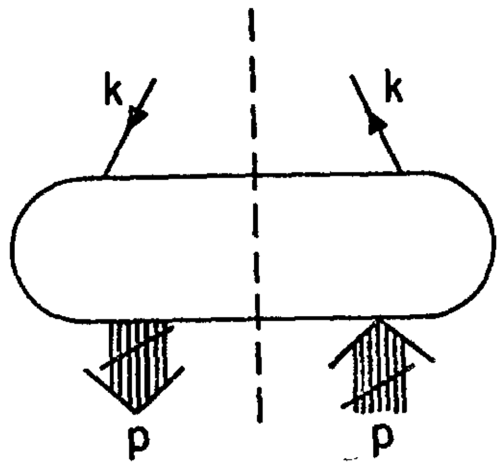


Fig. 2

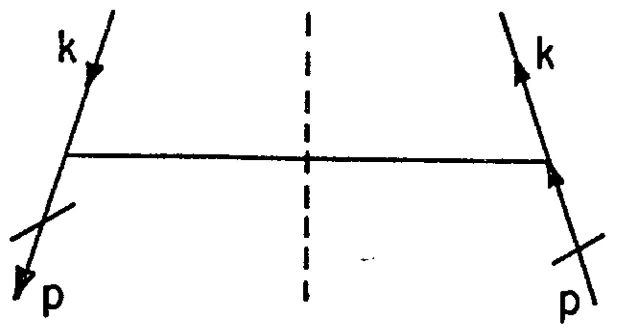


Fig. 3

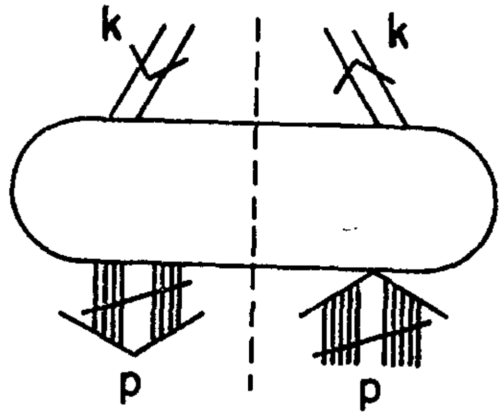


Fig. 4

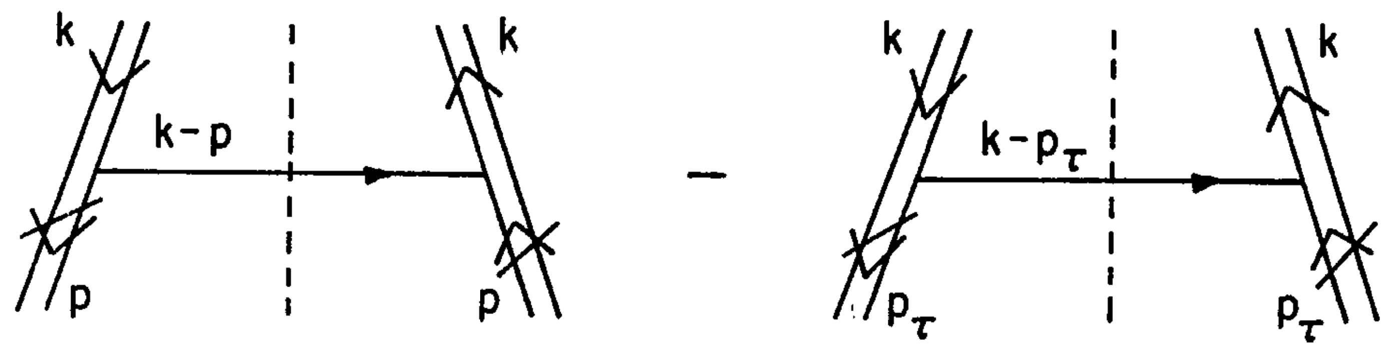
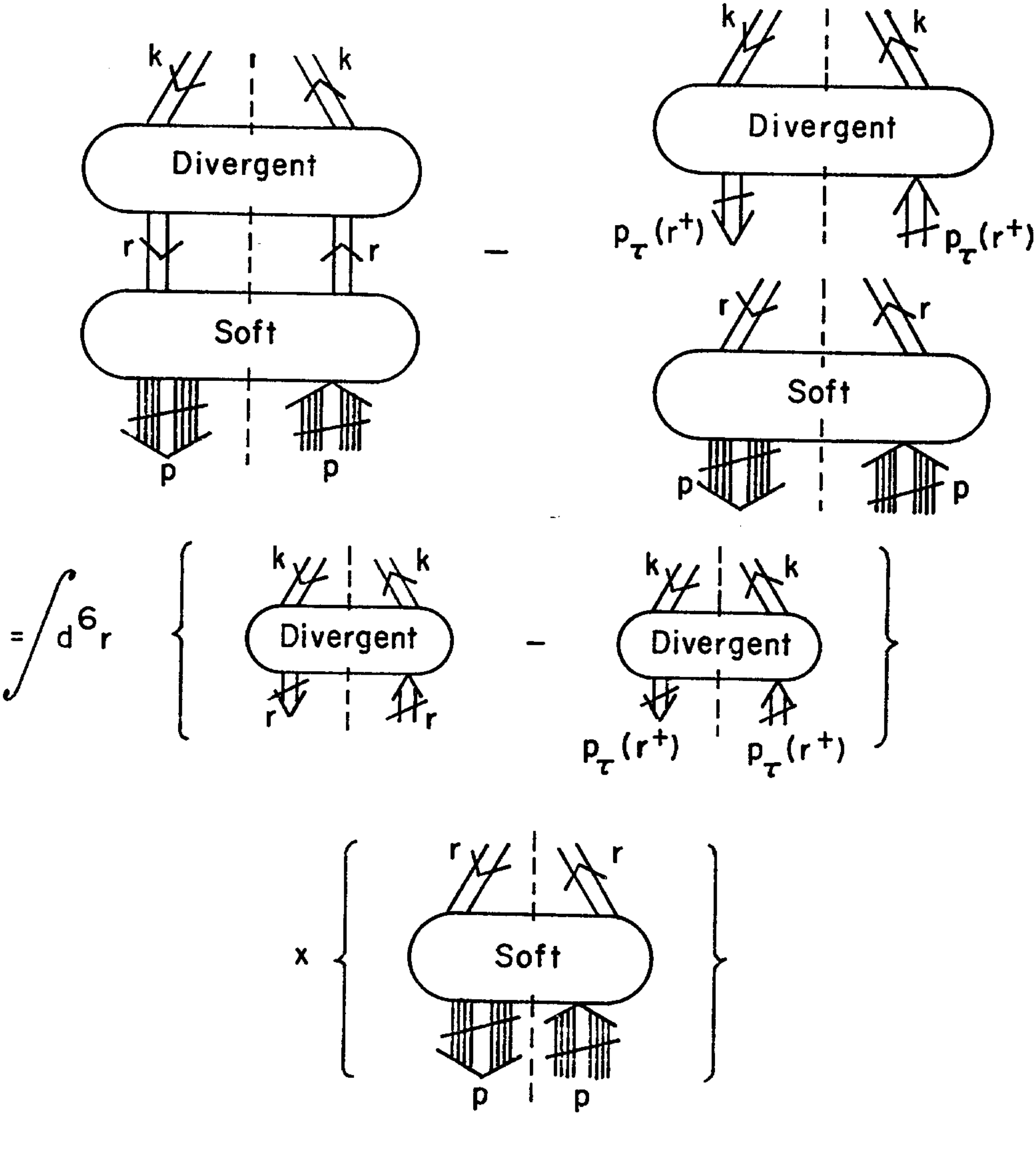


Fig. 5



k
k
Divergent
r
r
Soft
p
p
k
k
Divergent
$p_\tau(r^+)$
$p_\tau(r^+)$
r
r
Soft
p
p
$= \int d^6r$
k
k
Divergent
r
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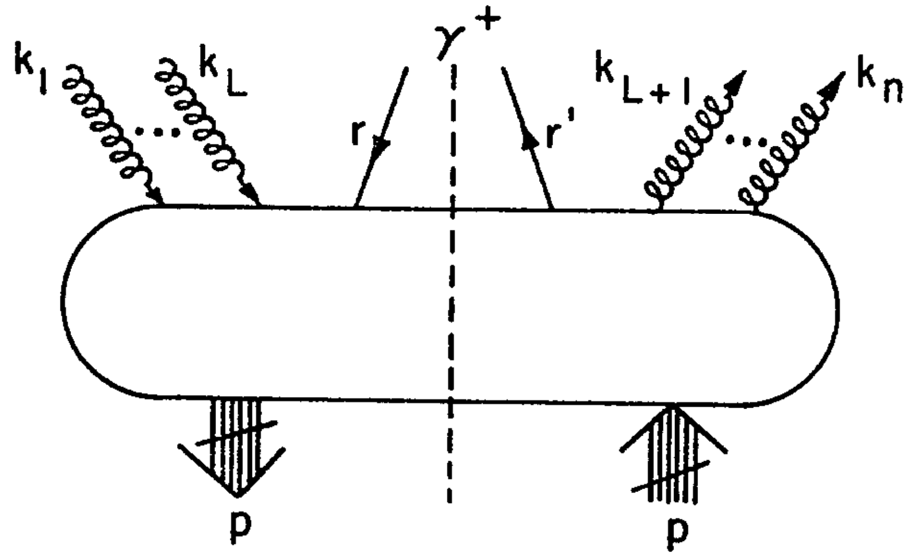
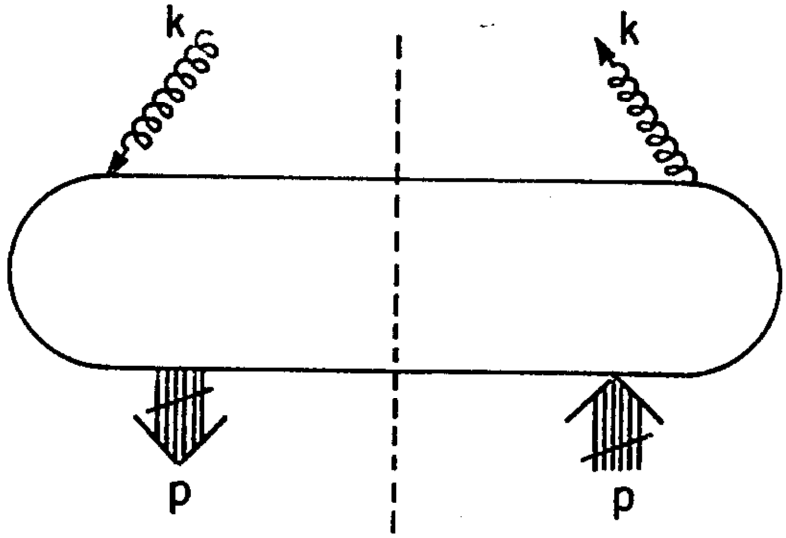
k
Divergent
$p_\tau(r^+)$
$p_\tau(r^+)$
x
r
r
Soft
p
p

Fig. 6



(a)



(b)

Fig. 7

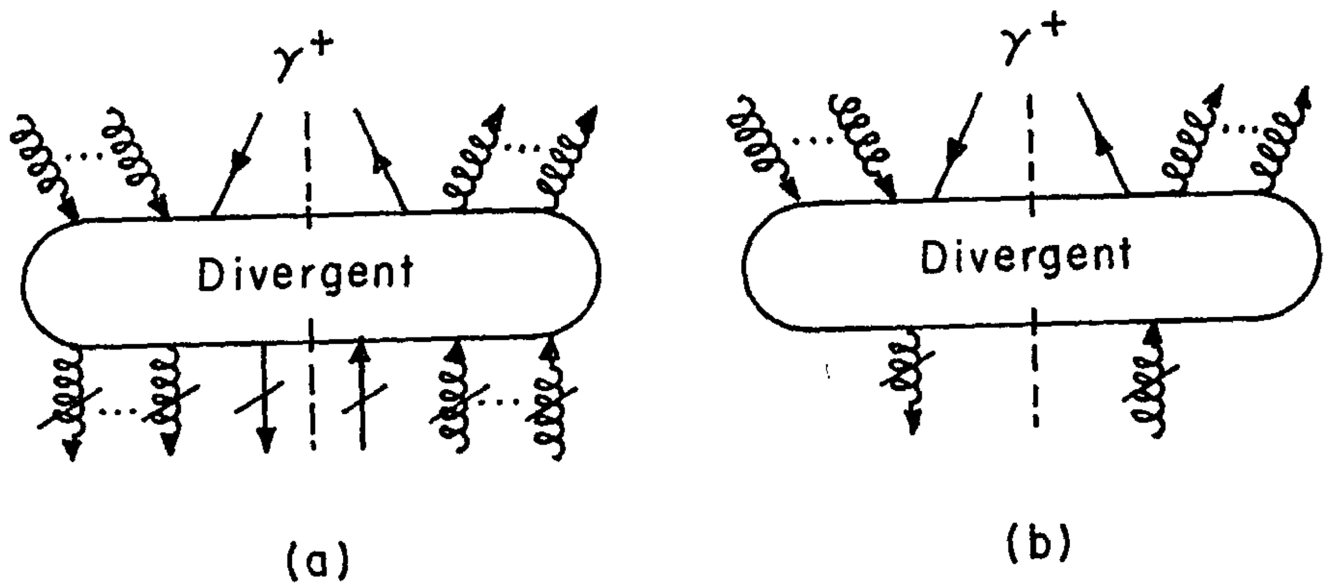


Fig. 8

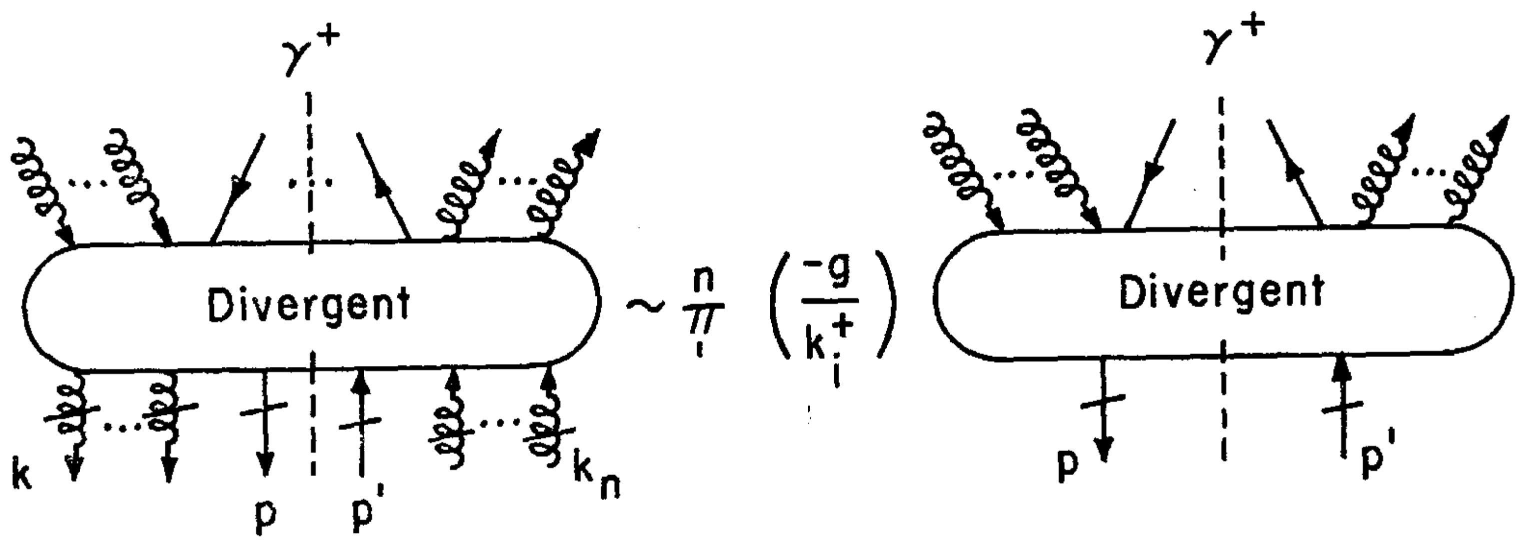


Fig. 9

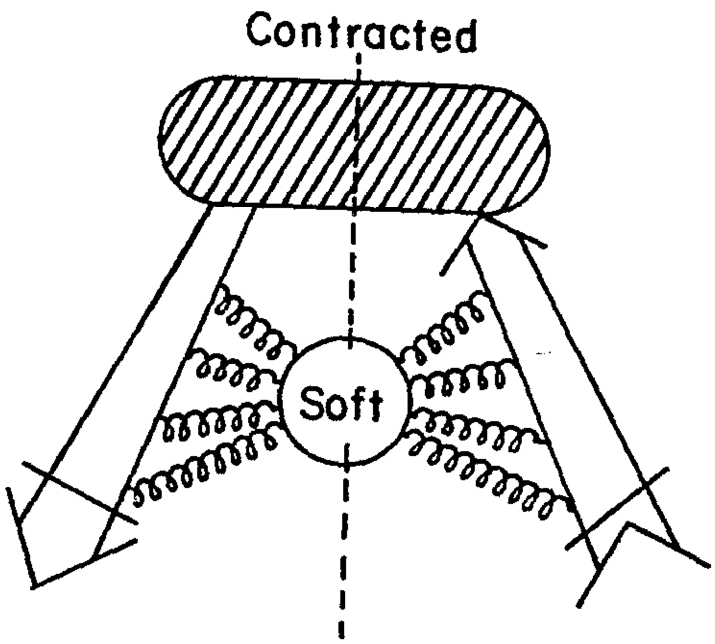


Fig. 10

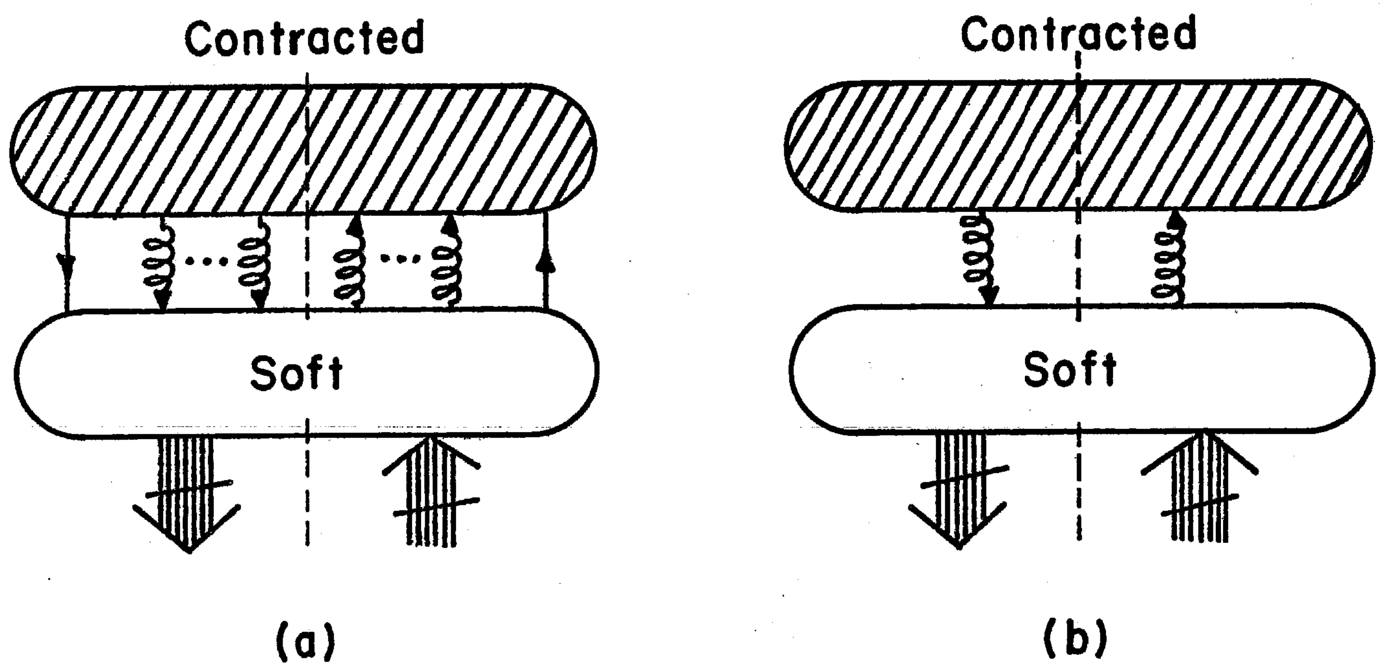


Fig. 11

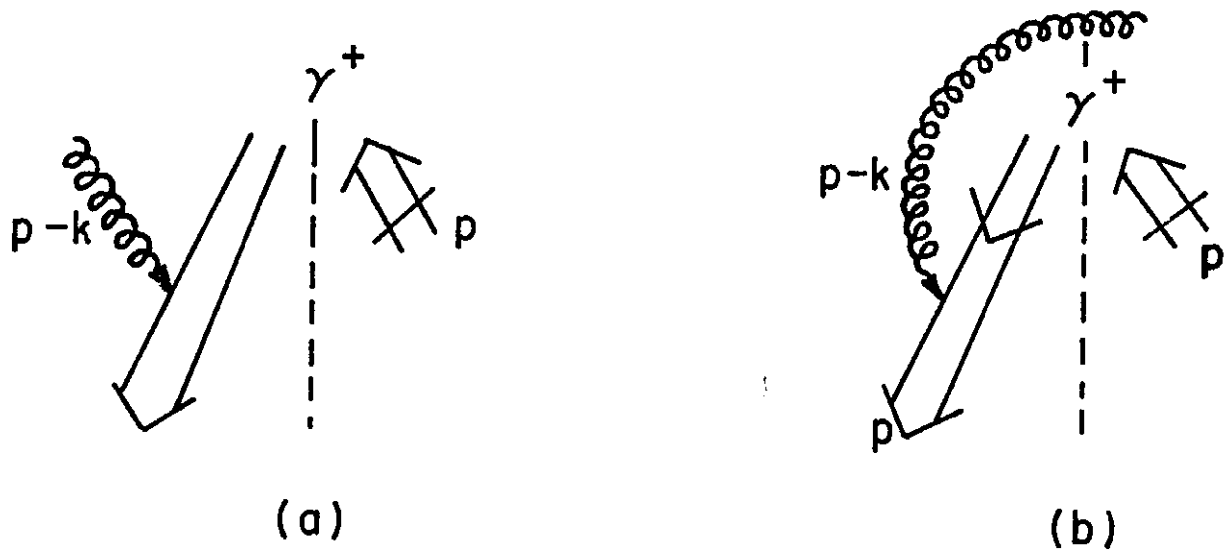
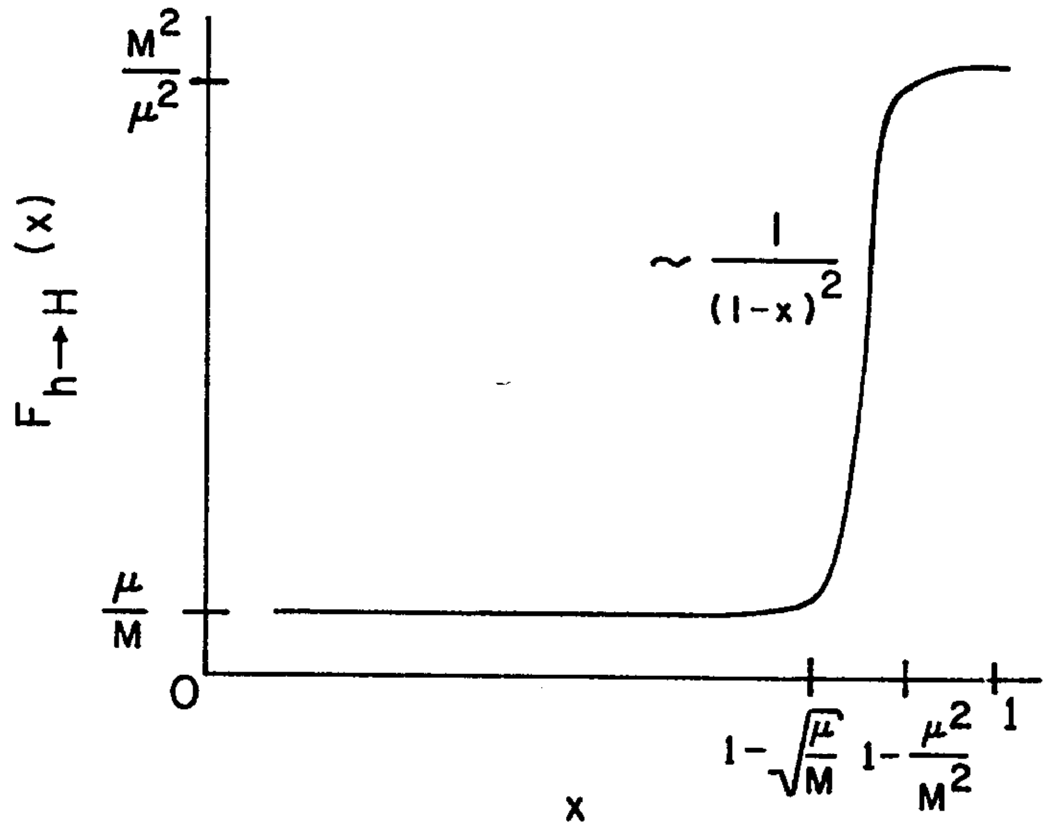


Fig. 12



$\frac{M^2}{\mu^2}$
$F_{h \to H}(x)$
$\sim \frac{1}{(1-x)^2}$
$\frac{\mu}{M}$
O
$1-\sqrt{\frac{\mu}{M}}$
$1-\frac{\mu^2}{M^2}$
1
x

Fig. 13

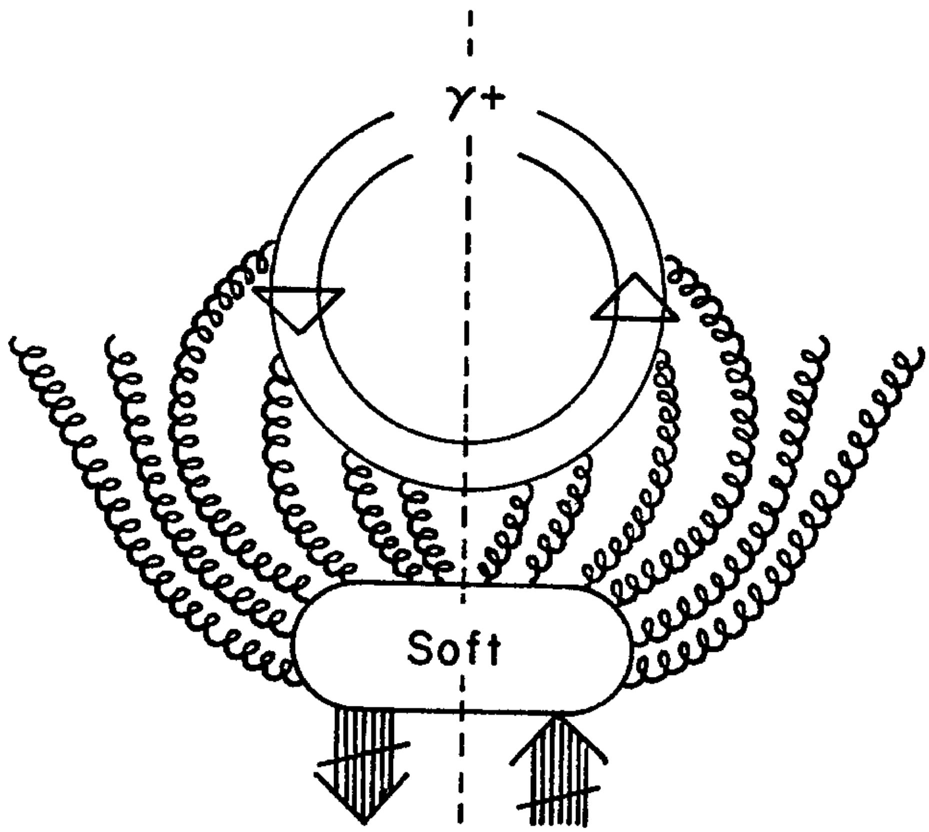


Fig. 14

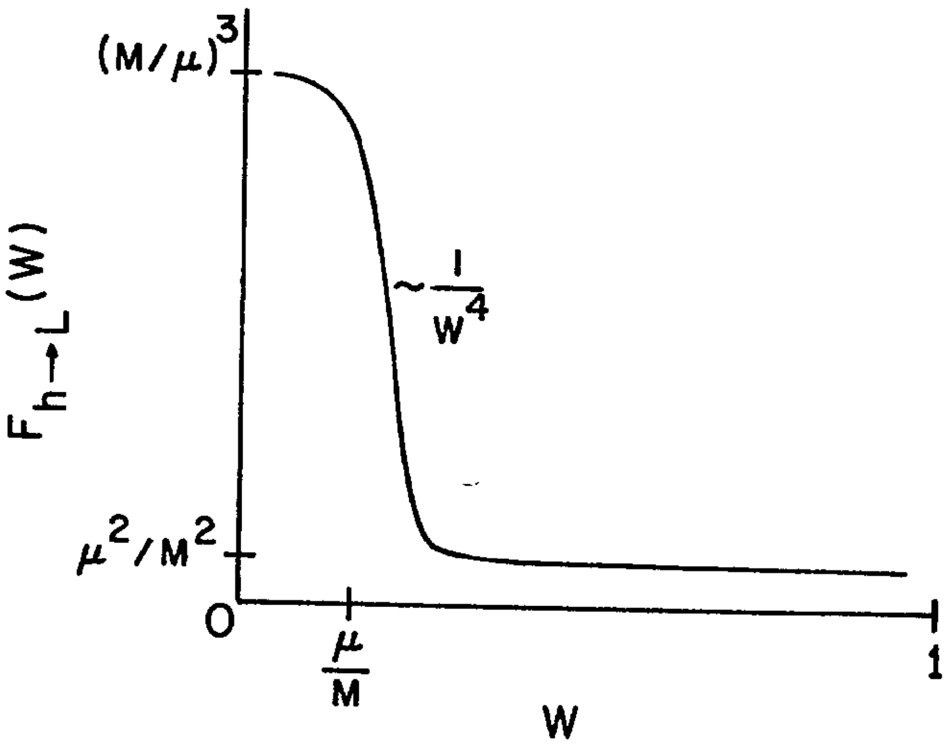


Fig. 15

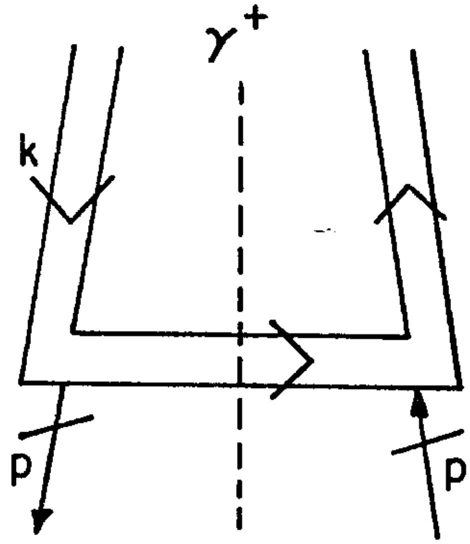


Fig. 16